RECIPROCAL SYSTEMS OF NON-ORTHOGONAL QUANTUM STATES

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Introduction. The elements of the systems of states encountered in quantum mechanical problems are typically orthogonal to one another, either because they have, in point of computational convenience, been *assumed* to be, or because it was as *eigenstates of an observable* that they were recommended to our attention. But when we construct the density matrix[1]

$$\boldsymbol{\rho} = \sum_{\text{ensemble}} |n)p_n(n| \tag{1}$$

to describe a statistical mixture of states there is, in general, no reason to assume $(m|n) = \delta_{mn}$; no principle of physics prevents our mixing *non*-orthogonal states.[2] I have had occasion elsewhere[3] to notice that because $\boldsymbol{\rho}$ is hermitian it possesses a population of real eigenvalues ρ_k and orthonormal (!) eigenvectors $|\rho_k)$, from which it acquires the spectral representation

$$\boldsymbol{\rho} = \sum_k |\rho_k)\rho_k(\rho_k| \tag{2}$$

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[1] According to Max Jammer, this device—usually attributed to John von Neumann—was also invented independently, and at the same time, by Hermann Weyl. For historical details see Jammer's very interesting Chapter 9 in *The Conceptual Development of Quantum Mechanics* (1966).

[2] This point of principle is not contradicted by the circumstance that in its most frequently encountered application the density matrix

$$\boldsymbol{\rho} = \sum_n |n)\tfrac{1}{Z}e^{-\frac{1}{kT}E_n}(n|$$

refers to a *thermalized mixture of energy eigenstates*, and those, of course, *are* orthonormal if the energy spectrum is non-degenerate.

[3] "Status and Ramifications of Ehrenfest's Theorem," (1998).

It can be shown that the numbers ρ_k are non-negative and sum to unity (because the numbers p_n are and do), and follows therefore from (2) that the mixture—originally represented to contain states $\{|n)\}$ with probabilities $\{p_n\}$—can as well be claimed to contain states $\{|\rho_k)\}$ with probabilities $\{\rho_k\}$. The "mixed state" concept is susceptible, therefore, to a certain fundamental ambituity; (2) stands at the "spectral center" of a *population of alternative conceptualizations* of the same root notion, and it is with the population (not with any of its arbitrarily selected individual members) that the physics of the matter is most properly associated.

Just as we might write $|\psi) \sim e^{i(\text{phase})}|\psi)$ to describe the ambiguity present in the concept of "state vector," i.e., to survey the population of state vectors physically equivalent to a given state vector $|\psi)$,[4] so would we like to be in position to *characterize the population* of alternative representations (1) of any given instance of (2). Each of those, since equivalent to the same spectral representation (2), is physically equivalent to each of the others, and each is distinguished from the spectral representation by the presence of some degree of non-orthogonality among the member states. We stand in evident need of tools adequate to permit the efficient management of non-orthogonality conditions, and it is apparent that the standard device—*get rid* of the non-orthogonality by adoption of some orthogonalization procedure—is not appropriate to the problem at hand, for it would in general do violence to the structure

$$\boldsymbol{\rho} = \text{weighted sum of } \textit{projection} \text{ operators}$$

characteristic of all density matrices.

1. Reciprocal sets in real vector spaces. Let vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$ span the real inner-product space $\mathcal{R}_n$, but be subject to no presumed inner-product relationships beyond the one

$$\begin{vmatrix} \boldsymbol{a}_1\cdot\boldsymbol{a}_1 & \boldsymbol{a}_1\cdot\boldsymbol{a}_2 & \ldots & \boldsymbol{a}_1\cdot\boldsymbol{a}_n \\ \boldsymbol{a}_2\cdot\boldsymbol{a}_1 & \boldsymbol{a}_2\cdot\boldsymbol{a}_2 & \ldots & \boldsymbol{a}_2\cdot\boldsymbol{a}_n \\ \vdots & \vdots & & \vdots \\ \boldsymbol{a}_n\cdot\boldsymbol{a}_1 & \boldsymbol{a}_n\cdot\boldsymbol{a}_2 & \ldots & \boldsymbol{a}_n\cdot\boldsymbol{a}_n \end{vmatrix} \neq 0 \tag{3}$$

implicit in the presumed linear independence of the vectors $\boldsymbol{a}_i$; in particular, we do not presume orthonormality:

$$\boldsymbol{a}_i\cdot\boldsymbol{a}_j = \delta_{ij} \quad \text{is } \textit{not} \text{ assumed}$$

An arbitrary vector $\boldsymbol{X} \in \mathcal{R}_n$ can be developed

$$\boldsymbol{X} = X^k\boldsymbol{a}_k \quad : \quad \text{summation convention understood}$$

[4] It is interesting that no such ambituity survives in the associated density matrix $\boldsymbol{\rho} \equiv |\psi)(\psi|$.

so we have

$$
\begin{aligned}
\boldsymbol{a}_j\!\cdot\!\boldsymbol{X} &= g_{jk}X^k \\
g_{jk} &\equiv \boldsymbol{a}_j\!\cdot\!\boldsymbol{a}_k
\end{aligned}
\tag{4}
$$

By standard convention $\mathbb{G} = \|g_{jk}\|$ and $\mathbb{G}^{-1} = \|g^{ij}\|$,[5] in which notation we have

$$
g^{ij}\boldsymbol{a}_j\!\cdot\!\boldsymbol{X} = g^{ij}g_{jk}X^k = X^i
$$

from which we obtain the decomposition formula

$$
\boldsymbol{X} = \underbrace{\boldsymbol{a}_i g^{ij}\boldsymbol{a}_j\!\cdot}_{\text{effectively the \textit{identity operator}}}\boldsymbol{X} \tag{5}
$$

If the set $\{\boldsymbol{a}_k\}$ does in fact possess the orthonormality property, then g^{ij} is 1 or 0 according as $i = j$ or $i \neq j$, and (5) assumes a form

$$
\boldsymbol{X} = \sum_i \boldsymbol{a}_i\, \boldsymbol{a}_i\!\cdot\!\boldsymbol{X}
$$

These last results fall much more familiarly upon the eye if one appropriates the essence of Dirac's notational trick, writing

$$
\begin{aligned}
|X) &= \sum_i\sum_j |a_i)g^{ij}(a_j|X) \\
&\downarrow \\
&= \sum_i |a_i)(a_i|X) \quad \text{in the orthonormal case}
\end{aligned}
$$

We conclude that the orthonormal *completeness condition* $\sum |a_i)(a_i| = \mathbb{I}$ should in the more general case be expressed

$$
\sum_i\sum_j |a_i)g^{ij}(a_j| = \mathbb{I} \tag{6}
$$

The question now presents itself: What—beyond the fact that they arise by inversion of $\mathbb{G} \equiv \|(a_i|a_j)\|$—can we say about the numbers g^{ij} which enter so critically into (6)? Introduce vectors $\{\boldsymbol{A}^1, \boldsymbol{A}^2, \ldots, \boldsymbol{A}^n\}$ which are "reciprocal" to $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$ in the defining sense that they satisfy the "biorthogonality condition"

$$
\boldsymbol{A}^i\!\cdot\!\boldsymbol{a}_j = \delta^i{}_j \quad \text{which in Dirac notation reads} \quad (A^i|a_j) = \delta^i{}_j \tag{7}
$$

[5] In this notation (3) reads $g \equiv \det\mathbb{G} \neq 0$, so the existence of $\mathbb{G}^{-1}$ is assured.

Immediately $(A^i|X) = (A^i|a_k)X^j = \delta^i{}_k X^k = X^i = g^{ik}(a_k|X)$ which, since valid for *all* $|X)$, entails

$$\left.\begin{aligned}(A^i| = g^{ij}(a_j| \text{ whence } |A^i) = g^{ij}|a_j)\ ;\textit{i.e.},\ \boldsymbol{A}^i &= g^{ij}\boldsymbol{a}_j\\ \boldsymbol{a}_j &= g_{jk}\boldsymbol{A}^k\end{aligned}\right\} \tag{8}$$

and in a particular case (set $\boldsymbol{X} = \boldsymbol{A}^j$) gives

$$g^{ij} = (A^i|A^j) \quad \Longleftarrow \text{compare} \Longrightarrow \quad g_{ij} = (a_i|a_j) \tag{9}$$

Equations (7–9) make especially clear the sense in which the spanning sets $\{\boldsymbol{a}_k\}$ and $\{\boldsymbol{A}^k\}$ are "reciprocal."

2. Explicit construction of the reciprocal set in the 2-dimensional case. It is in this simplest case feasible to proceed directly from (8). From

$$\mathbb{G} = \begin{vmatrix} \boldsymbol{a}_1\cdot\boldsymbol{a}_1 & \boldsymbol{a}_1\cdot\boldsymbol{a}_2 \\ \boldsymbol{a}_2\cdot\boldsymbol{a}_1 & \boldsymbol{a}_2\cdot\boldsymbol{a}_2 \end{vmatrix}$$

we obtain

$$\mathbb{G}^{-1} = \|g^{ij}\| = (1/g)\begin{vmatrix} \boldsymbol{a}_2\cdot\boldsymbol{a}_2 & -\boldsymbol{a}_1\cdot\boldsymbol{a}_2 \\ -\boldsymbol{a}_2\cdot\boldsymbol{a}_1 & \boldsymbol{a}_1\cdot\boldsymbol{a}_1 \end{vmatrix} \tag{10.1}$$

with

$$\begin{aligned} g \equiv \det\mathbb{G} &= a_1^2 a_2^2 - (\boldsymbol{a}_1\cdot\boldsymbol{a}_2)^2 > 0 \text{ by Schwarz' inequality} \\ &= (a_1 a_2 \sin\theta)^2 = (\text{area of parallelogram})^2 \end{aligned} \tag{10.2}$$

where

$$\begin{aligned} a_1 &\equiv \text{length of } \boldsymbol{a}_1 \\ a_2 &\equiv \text{length of } \boldsymbol{a}_2 \\ \theta &\equiv \text{angle subtended between } \boldsymbol{a}_1 \text{ and } \boldsymbol{a}_1 \end{aligned}$$

Returning with (10) to (8), we have

$$\left.\begin{aligned} \boldsymbol{A}^1 &= \tfrac{1}{g}\big\{(\boldsymbol{a}_2\cdot\boldsymbol{a}_2)\boldsymbol{a}_1 - (\boldsymbol{a}_1\cdot\boldsymbol{a}_2)\boldsymbol{a}_2\big\} \\ \boldsymbol{A}^2 &= \tfrac{1}{g}\big\{(\boldsymbol{a}_1\cdot\boldsymbol{a}_1)\boldsymbol{a}_2 - (\boldsymbol{a}_2\cdot\boldsymbol{a}_1)\boldsymbol{a}_1\big\} \end{aligned}\right\} \tag{11}$$

Quick calculation establishes that $\boldsymbol{A}^1\cdot\boldsymbol{a}_1 = \boldsymbol{A}^2\cdot\boldsymbol{a}_2 = 1$ and that

$$\boldsymbol{A}^1 \perp \boldsymbol{a}_2 \quad \text{and} \quad \boldsymbol{A}^2 \perp \boldsymbol{a}_1 \tag{12}$$

We observe finally that

$$\begin{aligned} \boldsymbol{A}^1\cdot\boldsymbol{A}^1 &= (1/g)(\text{length of } \boldsymbol{a}_2)^2 \\ \boldsymbol{A}^2\cdot\boldsymbol{A}^2 &= (1/g)(\text{length of } \boldsymbol{a}_1)^2 \\ \boldsymbol{A}^1\cdot\boldsymbol{A}^2 &= (1/g)(\boldsymbol{a}_1\cdot\boldsymbol{a}_2) \end{aligned}$$

and that

$$\text{physical dimension of } \boldsymbol{A} = \frac{1}{\text{physical dimension of } \boldsymbol{a}} \tag{13}$$

These last remarks make our use of the term "reciprocal" to describe the relationship of $\{\boldsymbol{A}^k\}$ to $\{\boldsymbol{a}_k\}$ seem all the more apt.

Alternatively, we might take the requirement (12) as our starting point. Quick calculation then establishes that necessarily

$$\begin{aligned}\boldsymbol{A}^1 &= \lambda_1\big\{(\boldsymbol{a}_2\cdot\boldsymbol{a}_2)\boldsymbol{a}_1 - (\boldsymbol{a}_1\cdot\boldsymbol{a}_2)\boldsymbol{a}_2\big\}\\ \boldsymbol{A}^2 &= \lambda_2\big\{(\boldsymbol{a}_1\cdot\boldsymbol{a}_1)\boldsymbol{a}_2 - (\boldsymbol{a}_2\cdot\boldsymbol{a}_1)\boldsymbol{a}_1\big\}\end{aligned}$$

and that to achieve $\boldsymbol{A}^1\cdot\boldsymbol{a}_1 = \boldsymbol{A}^2\cdot\boldsymbol{a}_2 = 1$ we must set

$$\lambda_1 = \lambda_2 = \frac{1}{(\boldsymbol{a}_1\cdot\boldsymbol{a}_1)(\boldsymbol{a}_2\cdot\boldsymbol{a}_2) - (\boldsymbol{a}_1\cdot\boldsymbol{a}_2)(\boldsymbol{a}_2\cdot\boldsymbol{a}_1)} = 1/g$$

and so we recover precisely (11), from which the numbers g^{ij} can (by appeal to (8)) simply be *read off*. It is this latter approach which, as will emerge, serves better to illuminate the general case.

3. Explicit construction of the reciprocal set in the 3-dimensional case. The obvious way to construct vectors $\{\boldsymbol{A}^1, \boldsymbol{A}^2, \boldsymbol{A}^3\}$ which are consistent with this generalization of (12)

$$\boldsymbol{A}^1 \perp \boldsymbol{a}_2 \;\&\; \boldsymbol{a}_3, \quad \boldsymbol{A}^2 \perp \boldsymbol{a}_3 \;\&\; \boldsymbol{a}_1 \quad \text{and} \quad \boldsymbol{A}^3 \perp \boldsymbol{a}_1 \;\&\; \boldsymbol{a}_2$$

is to write

$$\left.\begin{aligned}\boldsymbol{A}^1 &= \lambda_1\big\{\boldsymbol{a}_2\times\boldsymbol{a}_3\big\}\\ \boldsymbol{A}^2 &= \lambda_2\big\{\boldsymbol{a}_3\times\boldsymbol{a}_1\big\}\\ \boldsymbol{A}^3 &= \lambda_3\big\{\boldsymbol{a}_1\times\boldsymbol{a}_2\big\}\end{aligned}\right\} \tag{14.1}$$

To achieve $\boldsymbol{A}^1\cdot\boldsymbol{a}_1 = \boldsymbol{A}^2\cdot\boldsymbol{a}_2 = \boldsymbol{A}^2\cdot\boldsymbol{a}_2 = 1$ we have then only to set[6]

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda \equiv \frac{1}{(\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3)} = \frac{1}{\text{volume of parallelopiped}} \tag{14.2}$$

We observe that the dimensional relationship (13) is again enforced. With the aid of the elementary identity

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d}) &= (\boldsymbol{acd})\,\boldsymbol{b} - (\boldsymbol{bcd})\,\boldsymbol{a}\\ &= (\boldsymbol{abd})\,\boldsymbol{c} - (\boldsymbol{abc})\,\boldsymbol{d}\end{aligned}$$

[6] Here I borrow from H. Lass (*Vector and Tensor Analysis* (1950), p. 24)—who borrowed from E. B. Wilson's account (*Vector Analysis* (1901), p. 110) of the vector analysis of Gibbs—a useful yet not entirely standard notation for the "triple scalar product"

$$(\boldsymbol{abc}) \equiv \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$$

we find that

$$\begin{aligned}\text{volume of reciprocal parallelopiped} &= (\boldsymbol{A}^1\boldsymbol{A}^2\boldsymbol{A}^3)\\ &= \lambda^3(\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3)^2\\ &= \frac{1}{(\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3)}\\ &= \frac{1}{\text{volume of (direct) parallelopiped}}\end{aligned}$$

which lends further naturalness to our use of the term "reciprocal."

I note in passing that we have here converged upon ideas which have for a long time been standard to mathematical crystalographers, and have in more recent times become standard to solid state physics. The former tradition stems from work of Bravis,[7] and it was another Frenchman—Léon Brillouin—who was among the first to draw attention to the quantum mechanical utility of Bravis' ideas.[8] In such a context, an interest in non-orthogonal sets $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ is dictated by the *physical design* of crystals, which must be accepted as a fact of life from which no "orthogonalization procedure" can provide escape. Collateral interest in the associated reciprocal set arises in part from the form the exponential which enters into the definition of the Fourier transform.

Returning with the following elementary identity

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) = (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) - (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c})$$

to (14), and proceeding with (9) in mind, we compute

$$\begin{aligned}g^{11} &= \boldsymbol{A}^1\cdot\boldsymbol{A}^1 = \lambda^2\big\{(\boldsymbol{a}_2\cdot\boldsymbol{a}_2)(\boldsymbol{a}_3\cdot\boldsymbol{a}_3) - (\boldsymbol{a}_2\cdot\boldsymbol{a}_3)(\boldsymbol{a}_3\cdot\boldsymbol{a}_2)\big\}\\ g^{21} = g^{12} &= \boldsymbol{A}^1\cdot\boldsymbol{A}^2 = \lambda^2\big\{(\boldsymbol{a}_2\cdot\boldsymbol{a}_3)(\boldsymbol{a}_3\cdot\boldsymbol{a}_1) - (\boldsymbol{a}_2\cdot\boldsymbol{a}_1)(\boldsymbol{a}_3\cdot\boldsymbol{a}_3)\big\}\\ g^{31} = g^{13} &= \boldsymbol{A}^1\cdot\boldsymbol{A}^3 = \lambda^2\big\{(\boldsymbol{a}_2\cdot\boldsymbol{a}_1)(\boldsymbol{a}_3\cdot\boldsymbol{a}_2) - (\boldsymbol{a}_2\cdot\boldsymbol{a}_2)(\boldsymbol{a}_3\cdot\boldsymbol{a}_1)\big\}\\ g^{22} &= \boldsymbol{A}^2\cdot\boldsymbol{A}^2 = \lambda^2\big\{(\boldsymbol{a}_3\cdot\boldsymbol{a}_3)(\boldsymbol{a}_1\cdot\boldsymbol{a}_1) - (\boldsymbol{a}_3\cdot\boldsymbol{a}_1)(\boldsymbol{a}_1\cdot\boldsymbol{a}_3)\big\}\\ g^{32} = g^{23} &= \boldsymbol{A}^2\cdot\boldsymbol{A}^3 = \lambda^2\big\{(\boldsymbol{a}_3\cdot\boldsymbol{a}_1)(\boldsymbol{a}_1\cdot\boldsymbol{a}_2) - (\boldsymbol{a}_3\cdot\boldsymbol{a}_2)(\boldsymbol{a}_1\cdot\boldsymbol{a}_1)\big\}\\ g^{33} &= \boldsymbol{A}^3\cdot\boldsymbol{A}^3 = \lambda^2\big\{(\boldsymbol{a}_1\cdot\boldsymbol{a}_1)(\boldsymbol{a}_2\cdot\boldsymbol{a}_2) - (\boldsymbol{a}_1\cdot\boldsymbol{a}_2)(\boldsymbol{a}_2\cdot\boldsymbol{a}_1)\big\}\end{aligned}$$

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[7] August Bravis (1811–1863) was a French naval officer and adventurer (he climbed Mont Blanc and other major peaks, participated in the exploration of Lapland, etc.) who made significant contributions to a remarkable variety of scientific disciplines. It was in 1848 that he described the 14 possible regular arrangements of points in 3-space (that classic paper was reprinted in English translation as Memoir N° 1 by the Crystalographic Society of America in 1949); his ideas were further elaborated in his posthumous *Études cristallographiques* (1866).

[8] For a good account of the material to which I allude, see Chapters 4–7 of N. W. Ashcroft & N. D. Mermin, *Solid State Physics* (1976), where precisely my equations (14) can be found on p. 86. See also Chapter 6 in Brillouin's *Wave Propagation in Periodic Structures* (2nd edition, 1946).

In other words,

$$\mathbb{G}^{-1} = \lambda^2 \begin{pmatrix} +\begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{vmatrix} & -\begin{vmatrix} g_{21} & g_{23} \\ g_{31} & g_{33} \end{vmatrix} & +\begin{vmatrix} g_{21} & g_{22} \\ g_{31} & g_{32} \end{vmatrix} \\ -\begin{vmatrix} g_{12} & g_{13} \\ g_{32} & g_{33} \end{vmatrix} & +\begin{vmatrix} g_{11} & g_{13} \\ g_{31} & g_{33} \end{vmatrix} & -\begin{vmatrix} g_{11} & g_{12} \\ g_{31} & g_{32} \end{vmatrix} \\ +\begin{vmatrix} g_{12} & g_{13} \\ g_{22} & g_{23} \end{vmatrix} & -\begin{vmatrix} g_{11} & g_{13} \\ g_{21} & g_{23} \end{vmatrix} & +\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} \end{pmatrix}$$

But this—provided we can establish that

$$\lambda^2 = \frac{1}{\det \mathbb{G}} \tag{15}$$

—is precisely what we would have written down had we set out to *compute* $\mathbb{G}^{-1}$ by means of the standard matrix inversiton algorithm.

The proof of (15), though not difficult, is in fact quite informative, but requires some notational preparation: let us for the moment agree—the better to keep simple things simple, and to avoid the distraction of a bewildering profusion of indices—to write

$$\boldsymbol{a} \text{ for } \boldsymbol{a}_1,\ \boldsymbol{b} \text{ for } \boldsymbol{a}_2,\ \boldsymbol{c} \text{ for } \boldsymbol{a}_3,\ \boldsymbol{A} \text{ for } \boldsymbol{A}^1,\ \boldsymbol{B} \text{ for } \boldsymbol{A}^2,\ \boldsymbol{C} \text{ for } \boldsymbol{A}^3$$

and proceeding in reference to some/any orthonormal basis

$$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\} \quad \text{with} \quad \boldsymbol{e}_i\cdot\boldsymbol{e}_j = \delta_{ij}$$

Let us agree, moreover, to write $\boldsymbol{a} = \sum \tilde{a}^i \boldsymbol{e}_i$, etc. and in that sense to understand the standard notations

$$\boldsymbol{a} = \begin{pmatrix} \tilde{a}^1 \\ \tilde{a}^2 \\ \tilde{a}^3 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} \tilde{b}^1 \\ \tilde{b}^2 \\ \tilde{b}^3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{c} = \begin{pmatrix} \tilde{c}^1 \\ \tilde{c}^2 \\ \tilde{c}^3 \end{pmatrix}$$

To obtain (15) we have only to notice that

$$\begin{pmatrix} \tilde{a}^1 & \tilde{a}^2 & \tilde{a}^3 \\ \tilde{b}^1 & \tilde{b}^2 & \tilde{b}^3 \\ \tilde{c}^1 & \tilde{c}^2 & \tilde{c}^3 \end{pmatrix} \begin{pmatrix} \tilde{a}^1 & \tilde{b}^1 & \tilde{c}^1 \\ \tilde{a}^2 & \tilde{b}^2 & \tilde{c}^2 \\ \tilde{a}^3 & \tilde{b}^3 & \tilde{c}^3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{a\cdot a} & \boldsymbol{a\cdot b} & \boldsymbol{a\cdot c} \\ \boldsymbol{b\cdot a} & \boldsymbol{b\cdot b} & \boldsymbol{b\cdot c} \\ \boldsymbol{c\cdot a} & \boldsymbol{c\cdot b} & \boldsymbol{c\cdot c} \end{pmatrix} = \mathbb{G}$$

Immediately

$$g \equiv \det \mathbb{G} = \begin{vmatrix} \tilde{a}^1 & \tilde{b}^1 & \tilde{c}^1 \\ \tilde{a}^2 & \tilde{b}^2 & \tilde{c}^2 \\ \tilde{a}^3 & \tilde{b}^3 & \tilde{c}^3 \end{vmatrix}^2 = (\boldsymbol{a\,b\,c})^2 \tag{16}$$

which establishes (15) and at the same time provides a seldom-encountered *coordinate-free description of the triple scalar product*:

$$(\boldsymbol{a\,b\,c}) \equiv \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \sqrt{\det \begin{pmatrix} \boldsymbol{a\cdot a} & \boldsymbol{a\cdot b} & \boldsymbol{a\cdot c} \\ \boldsymbol{b\cdot a} & \boldsymbol{b\cdot b} & \boldsymbol{b\cdot c} \\ \boldsymbol{c\cdot a} & \boldsymbol{c\cdot b} & \boldsymbol{c\cdot c} \end{pmatrix}} \tag{17}$$

Interestingly, one could use (17) to *assign* meaning to a "generalized triple scalar product" which makes sense—and exhibits all the familiar symmetry properties[9]—even when the vectors in question are not 3-vectors.[10] And by natural extension one could assign meaning to a

$$\begin{aligned} (\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{d}) &\quad:\quad \text{quadruple scalar product of } n\text{-vectors} \\ (\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{d}\,\boldsymbol{e}) &\quad:\quad \text{quintuple scalar product of } n\text{-vectors, etc.} \end{aligned}$$

The mechanism that lies at the base of (16) is at once simpler and deeper than I have represented it to be, and it is clear understanding of this fact that points the way toward the dimensional generalization of (14). Let an arbitrary vector $\boldsymbol{X}$ have coordinates X^i with respect to the generally non-orthogonal basis

$$\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\} \quad \text{with} \quad \boldsymbol{a}_i\cdot\boldsymbol{a}_j = g_{ij}$$

but coordinates $\tilde{X}^i$ with respect to the orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$: from $\boldsymbol{X} = X^i\boldsymbol{a}_i = X^i a^j{}_i\,\boldsymbol{e}_j = \tilde{X}^j\boldsymbol{e}_j$ we read

$$\begin{aligned} \begin{pmatrix} \tilde{X}^1 \\ \tilde{X}^2 \\ \tilde{X}^3 \end{pmatrix} &= \underbrace{\begin{pmatrix} a^1{}_1 & a^1{}_2 & a^1{}_3 \\ a^2{}_1 & a^2{}_2 & a^2{}_3 \\ a^3{}_1 & a^3{}_2 & a^3{}_3 \end{pmatrix}} \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix} \\ &= \begin{pmatrix} \tilde{a}^1 & \tilde{b}^1 & \tilde{c}^1 \\ \tilde{a}^2 & \tilde{b}^2 & \tilde{c}^2 \\ \tilde{a}^3 & \tilde{b}^3 & \tilde{c}^3 \end{pmatrix} \equiv \mathbb{A} \quad:\quad \text{transformation matrix} \end{aligned}$$

Evidently the expression which first announced itself in the form $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ has deeper significance as the *determinant of a transformation matrix*:

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = A \equiv \det\mathbb{A} \tag{18}$$

Since g_{ij} transforms covariantly, we have

$$g_{ij} = a^k{}_i a^l{}_j \tilde{g}_{kl} \quad \text{with} \quad \tilde{g}_{kl} = \delta_{kl}$$

giving

$$\begin{aligned} g = A^2\tilde{g} &\quad:\quad g \text{ transforms as a } \textit{density of weight } W = 2 \\ \tilde{g} = 1 & \end{aligned}$$

This, I claim, is the deeper—and readily generalizable—meaning of (16).

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[9] Those are simply the symmetries of ϵ^{ijk}, which is to say: the symmetries of the alternating group on three objects.

[10] Of course, when $n \neq 3$ it becomes meaningless to write $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$, and incorrect to write $(\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}) = \sqrt{\det\mathbb{G}}$.

We are in position now to reexpress (14)

$$\begin{aligned}
\tilde{A}^i &= \delta^{ij}\tilde{A}_j = \tilde{A}_i \quad \text{with} \quad \tilde{A}_i = \tfrac{1}{(\boldsymbol{abc})}\epsilon_{ijk}\tilde{b}^j\tilde{c}^k \\
\tilde{B}^i &= \delta^{ij}\tilde{B}_j = \tilde{B}_i \quad \text{with} \quad \tilde{B}_i = \tfrac{1}{(\boldsymbol{abc})}\epsilon_{ijk}\tilde{c}^j\tilde{a}^k \\
\tilde{C}^i &= \delta^{ij}\tilde{C}_j = \tilde{C}_i \quad \text{with} \quad \tilde{C}_i = \tfrac{1}{(\boldsymbol{abc})}\epsilon_{ijk}\tilde{a}^j\tilde{b}^k
\end{aligned}$$

$$(\boldsymbol{abc}) \equiv \epsilon_{pqr}\tilde{a}^p\tilde{b}^q\tilde{c}^r$$

Here as previously, the distracting tildes identify coordinates relative to an imported *orthonormal* basis, and the equations on the left remind us that, with respect to the metric δ_{ij}, index placement expresses a "distinction without a difference." But the preceding equations make such tensor-theoretic good sense as (with one obvious modification) to work in *any* coordinate system. If, in particular, we—as previously—take

$$\boldsymbol{a} = \begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{c} = \begin{pmatrix} c^1 \\ c^2 \\ c^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

to refer to the $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ basis most natural to this discussion, we have

$$\left.\begin{aligned}
A_i &= \tfrac{1}{\{\boldsymbol{abc}\}}\epsilon_{ijk}b^jc^k \quad \text{and} \quad A^i = g^{ij}A_j \\
B_i &= \tfrac{1}{\{\boldsymbol{abc}\}}\epsilon_{ijk}c^ja^k \quad \text{and} \quad B^i = g^{ij}B_j \\
C_i &= \tfrac{1}{\{\boldsymbol{abc}\}}\epsilon_{ijk}a^jb^k \quad \text{and} \quad C^i = g^{ij}C_j
\end{aligned}\right\} \tag{19}$$

$$\{\boldsymbol{abc}\} \equiv \epsilon_{pqr}a^pb^qc^r = 1$$

Recalling[11] that the Levi-Civita tensor transforms by numerical invariance if and only if transformed as a *tensor density of weight* $W = -1$, we see that (19) describes structures of the form

$$\frac{\text{vector density}}{\text{scalar density of same weight}} = \text{vector of zero weight}$$

Transparently

$$\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}\begin{pmatrix} a^1 & b^1 & c^1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Working from (19) we have

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

[11] See §3 in "Electrodynamical applications of the exterior calculus" (1996).

giving

$$\begin{pmatrix} A^1 \\ A^2 \\ A^3 \end{pmatrix} = 1^{\text{st}} \text{ column of } \mathbb{G}^{-1}$$
$$\begin{pmatrix} B^1 \\ B^2 \\ B^3 \end{pmatrix} = 2^{\text{nd}} \text{ column of } \mathbb{G}^{-1}$$
$$\begin{pmatrix} C^1 \\ C^2 \\ C^3 \end{pmatrix} = 3^{\text{rd}} \text{ column of } \mathbb{G}^{-1}$$

whence

$$\begin{aligned} \boldsymbol{A}^1 &= \tfrac{1}{g}\left\{ + \begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{vmatrix} \boldsymbol{a}_1 - \begin{vmatrix} g_{12} & g_{13} \\ g_{32} & g_{33} \end{vmatrix} \boldsymbol{a}_2 + \begin{vmatrix} g_{12} & g_{13} \\ g_{22} & g_{23} \end{vmatrix} \boldsymbol{a}_3 \right\} \\ \boldsymbol{A}^2 &= \tfrac{1}{g}\left\{ - \begin{vmatrix} g_{21} & g_{23} \\ g_{31} & g_{33} \end{vmatrix} \boldsymbol{a}_1 + \begin{vmatrix} g_{11} & g_{13} \\ g_{31} & g_{33} \end{vmatrix} \boldsymbol{a}_2 - \begin{vmatrix} g_{11} & g_{13} \\ g_{21} & g_{23} \end{vmatrix} \boldsymbol{a}_3 \right\} \\ \boldsymbol{A}^3 &= \tfrac{1}{g}\left\{ + \begin{vmatrix} g_{21} & g_{22} \\ g_{31} & g_{32} \end{vmatrix} \boldsymbol{a}_1 - \begin{vmatrix} g_{11} & g_{12} \\ g_{31} & g_{32} \end{vmatrix} \boldsymbol{a}_2 + \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} \boldsymbol{a}_3 \right\} \end{aligned}$$

at which point we have in effect recovered an instance of (8).

4. Explicit construction of the reciprocal set in the general case. Returning now to the generality of §1, let $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$ refer to any basis in $\mathcal{R}_n$, and write $\mathbb{G} \equiv \|g_{ij}\|$ with $g_{ij} \equiv \boldsymbol{a}_i \cdot \boldsymbol{a}_j$. In *component* form the elements of the reciprocal basis $\{\boldsymbol{A}^p : p = 1, 2, \ldots n\}$ can, in generalization of (19), be described

$$A^{pi} = g^{ij} A^p{}_j \quad \text{with} \quad A^p{}_j \equiv \tfrac{1}{A} \underbrace{\epsilon_{k_1 k_2 \ldots k_p \ldots k_n}}_{\uparrow \text{ replace with } j} a_1^{k_1} a_2^{k_2} \cdots \underbrace{a_p^{k_p}}_{\uparrow \text{ omit this factor}} \cdots a_n^{k_n} \tag{20}$$

with $A \equiv \epsilon_{k_1 k_2 \ldots k_n} a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}$. Here the k-indexed numbers a_q^k comprise the components of $\boldsymbol{a}_q$ relative to any selected basis (which *may* be the natural basis $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$, and *may* be some orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$, but need not be either one). Immediately,

$$A^p{}_j a_q^j = \delta^p{}_q \tag{21.1}$$

To say the same thing another way: if $\boldsymbol{A}^p = A^{pi} \boldsymbol{a}_i$ (here we have opted to work in the "natural basis," where $a_q^k = \delta^k{}_q$ causes (20) to simplify greatly) then

$$\begin{aligned} \boldsymbol{A}^p \cdot \boldsymbol{a}_q = A^{pi} g_{iq} = A^p{}_j \delta^j{}_q &= A^p{}_q \\ &= \frac{1}{\epsilon_{123\ldots n}} \underbrace{\epsilon_{12\ldots p \ldots n}}_{\uparrow \text{ replace with } q} = \delta^p{}_q \end{aligned} \tag{21.2}$$

which reproduces (7). These results demonstrate that (20) works, and in fact does its work fairly efficiently. It works, however, by appeal to a coordinate system. Means to avoid that formal defect—and thus to recover one of the more attractive features both of (11) and of (14)—are afforded by the exterior calculus.

Let $\boldsymbol{A}$ be an n-dimensional antisymmetric tensor of rank p, let $\boldsymbol{B}$ be ditto of rank q

$$\begin{aligned}\boldsymbol{A} &\prec A^{i_1 i_2 \dots i_p}\\ \boldsymbol{B} &\prec B^{j_1 j_2 \dots j_q}\end{aligned}$$

The "wedge product" of $\boldsymbol{A}$ and $\boldsymbol{B}$ (sometimes called their "exterior product") is defined[12]

$$\begin{aligned}\boldsymbol{A}\wedge\boldsymbol{B} &\prec \begin{cases}\frac{1}{p!q!}\delta^{i_1 i_2 \dots i_{p+q}}{}_{a_1 a_2 \dots a_p b_1 b_2 \dots b_q} A^{a_1 a_2 \dots a_p} B^{b_1 b_2 \dots b_q} & : \;p+q\le n\\ 0 & : \;p+q>n\end{cases}\\ &= \text{antisymmetrized tensor product}\end{aligned}$$

and has these notable properties:

$$\boldsymbol{A}\wedge(\boldsymbol{B}+\boldsymbol{C}) = (\boldsymbol{A}\wedge\boldsymbol{B}) + (\boldsymbol{A}\wedge\boldsymbol{C}) \;:\; \text{DISTRIBUTIVITY} \tag{22.1}$$

$$(\boldsymbol{A}\wedge\boldsymbol{B})\wedge\boldsymbol{C} = \boldsymbol{A}\wedge(\boldsymbol{B}\wedge\boldsymbol{C}) \;:\; \text{ASSOCIATIVITY} \tag{22.2}$$

$$\boldsymbol{A}\wedge\boldsymbol{B} = (-)^{pq}\boldsymbol{B}\wedge\boldsymbol{A} = \begin{cases}-\boldsymbol{B}\wedge\boldsymbol{A} & \text{if } p \text{ and } q \text{ are both odd}\\ +\boldsymbol{B}\wedge\boldsymbol{A} & \text{otherwise}\end{cases} \tag{22.3}$$

If, in particular, $\boldsymbol{A}$ and $\boldsymbol{B}$ are *vectors* (i.e., if $p=q=1$ and $n\ge 2$) then

$$\begin{aligned}\boldsymbol{A}\wedge\boldsymbol{B} \prec\; &\delta^{ij}{}_{ab}A^a B^b = A^i B^j - A^j B^i\\ &\uparrow\\ &\delta^{ij}{}_{ab} \equiv \begin{vmatrix}\delta^i{}_a & \delta^i{}_b\\ \delta^j{}_a & \delta^j{}_b\end{vmatrix} = \delta^i{}_a\delta^j{}_b - \delta^j{}_a\delta^i{}_b\end{aligned}$$

which generalizes properties familiarly associated (case $n=3$) with the cross product. From a *set* of vectors $\boldsymbol{A}_1, \boldsymbol{A}_2, \dots, \boldsymbol{A}_m$ $(m\le n)$ we can form this antisymmetric tensor of rank m

$$\boldsymbol{A}_1\wedge\boldsymbol{A}_2\wedge\cdots\wedge\boldsymbol{A}_m \prec\; \delta^{i_1 i_2 \dots i_m}{}_{a_1 a_2 \dots a_m} A_1^{a_1} A_2^{a_2}\cdots A_m^{a_m}$$

which in consequence of (22) vanishes unless the vectors in question are linearly independent.

---

[12] See my ELECTRODYNAMICS (1972), p. 152; H. Flanders, *Differential Forms, with Applications to the Physical Sciences* (1963), §2.3. Here I allow myself to borrow casually from the exterior calculus; all missing details can be found either in those sources or in the essay cited in the preceding footnote.

The vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ provice, by assumption, a "full house" of such vectors, and when wedged together yield an antisymmetric tensor with the special property that

$$\text{rank} = \text{dimension}$$

Such an object is the dual of a scalar, and the geometrical meaning of that scalar becomes obvious when one introduces the identity[13]

$$\delta^{i_1 i_2 \ldots i_n}{}_{j_1 j_2 \ldots j_n} = \varepsilon^{i_1 i_2 \ldots i_n} \epsilon_{j_1 j_2 \ldots j_n}$$

into

$$\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n \prec \ \delta^{i_1 i_2 \ldots i_n}{}_{j_1 j_2 \ldots j_n} a_1^{j_1} a_2^{j_2} \cdots a_n^{j_n}$$

Immediately

$$= \varepsilon^{i_1 i_2 \ldots i_n} \cdot \det \underbrace{\begin{pmatrix} a_1^1 & a_2^1 & \ldots & a_n^1 \\ a_1^2 & a_2^2 & \ldots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^n & a_2^n & \ldots & a_n^n \end{pmatrix}}_{\equiv \mathbb{A}} \tag{23}$$

The determinant is most familiar as the "Jacobian" encountered in connection with the transformation $X^i \mapsto \tilde{X}^i = a^i_j X^j$; it permits one to write (for example)

$$d\tilde{X}^1 d\tilde{X}^2 \cdots d\tilde{X}^1 = \underbrace{\frac{\partial(\tilde{X}^1, \tilde{X}^2, \ldots, \tilde{X}^n)}{\partial(X^1, X^2, \ldots, X^n)}}_{\text{Jacobian} = A \equiv \det \mathbb{A}} dX^1 dX^2 \cdots dX^n$$

---

[13] It is, in this connection, important to know that the Levi-Civita tensor comes actually in two flavors (which I attempt to distinguish notationally by using what TEX calls `\varepsilon` for the one, `\epsilon` for the other); one is contravariant, the other covariant, and each is assigned such weight as to cause it to *transform by numerical invariance*:

$$\varepsilon^{i_1 i_2 \ldots i_n} \equiv \operatorname{sgn}\begin{pmatrix} i_1 & i_2 & \ldots & i_n \\ 1 & 2 & \ldots & n \end{pmatrix} \ : \ \text{contravariant, of weight } W = +1$$

$$\epsilon_{j_1 j_2 \ldots j_n} \equiv \operatorname{sgn}\begin{pmatrix} 1 & 2 & \ldots & n \\ j_1 & j_2 & \ldots & j_n \end{pmatrix} \ : \ \text{covariant, of weight } W = -1$$

In metrically connected contexts (i.e., when tensors g_{ij} and g^{ij} are available to manipulate indices) one can also form $\varepsilon_{i_1 i_2 \ldots i_n}$ and $\epsilon^{j_1 j_2 \ldots j_n}$ which (since weight is no longer correctly mated to rank) do *not* transform by numerical invariance; one has

$$\varepsilon^{i_1 i_2 \ldots i_n} = g \cdot \epsilon^{i_1 i_2 \ldots i_n}$$

where (since g has weight $W = +2$) the weight of the expression on the left is the same as the net weight of the expression on the right of the equality. The expression on the left—by contrivance—transforms by numerical invariance, but g doesn't, so $\epsilon^{i_1 i_2 \ldots i_n}$ can't; its values range on $\{-g^{-1}, 0, +g^{-1}\}$, while those of $\varepsilon_{j_1 j_2 \ldots j_n}$ range on $\{-g, 0, +g\}$.

and can—as already at (18)—be interpreted as the volume of a parallelopiped.[14] From (23) if follows finally that

$$(\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n)^{\text{dual}} \prec \tfrac{1}{n!}\epsilon_{i_1 i_2 \ldots i_n}\big\{\varepsilon^{i_1 i_2 \ldots i_n} \cdot \det \mathbb{A}\big\} = A$$

The construction

$$\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \cancel{\boldsymbol{a}}_p \wedge \cdots \wedge \boldsymbol{a}_n \quad : \quad \boldsymbol{a}_p \text{ omitted}$$

is by nature a totally antisymmetric tensor of rank $n-1$, the dual of a vector. From the general relations (22) it follows readily that

$$\boldsymbol{a}_q \wedge (\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \cancel{\boldsymbol{a}}_p \wedge \cdots \wedge \boldsymbol{a}_n) = \begin{cases} \boldsymbol{0} \quad \text{if } q \neq p, \text{ but in the alternative case} \\ (-)^{p-1}\, \boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_p \wedge \cdots \wedge \boldsymbol{a}_n \end{cases}$$

It becomes natural in the light of this observation to write

$$\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \cancel{\boldsymbol{a}}_p \wedge \cdots \wedge \boldsymbol{a}_n \prec \delta^{i_1 i_2 \ldots \cancel{i}_p \ldots i_n}{}_{j_1 j_2 \ldots \cancel{j}_p \ldots j_n}\, a_1^{j_1} a_2^{j_2} \ldots \cancel{a}_p^{j_p} \ldots a_n^{j_n}$$

and to notice that

$$\begin{aligned}
&(\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \cancel{\boldsymbol{a}}_p \wedge \cdots \wedge \boldsymbol{a}_n)^{\text{dual}} \\
&\qquad \prec \tfrac{1}{(n-1)!}\epsilon_{j k_1 \ldots k_{n-1}}\big\{\delta^{k_1 k_2 \ldots k_{n-1}}{}_{j_1 j_2 \ldots \cancel{j}_p \ldots j_n}\, a_1^{j_1} a_2^{j_2} \ldots \cancel{a}_p^{j_p} \ldots a_n^{j_n}\big\} \\
&\qquad = \epsilon_{j j_1 j_2 \ldots \cancel{j}_p \ldots j_n}\, a_1^{j_1} a_2^{j_2} \ldots \cancel{a}_p^{j_p} \ldots a_n^{j_n} \\
&\qquad = (-)^{p-1} \underbrace{\epsilon_{k_1 k_2 \ldots k_p \ldots k_n}}_{\text{replace with } j}\, a_1^{k_1} a_2^{k_2} \cdots \underbrace{a_p^{k_p}}_{\text{omit this factor}} \cdots a_n^{k_n}
\end{aligned}$$

Evidently we have only to define[15]

$$\boldsymbol{M}^p \equiv (-)^{p-1} \frac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \cancel{\boldsymbol{a}}_p \wedge \cdots \wedge \boldsymbol{a}_n}{(\boldsymbol{a}_1 \boldsymbol{a}_2 \cdots \boldsymbol{a}_n)} \tag{24.1}$$

$$(\boldsymbol{a}_1 \boldsymbol{a}_2 \cdots \boldsymbol{a}_n) \equiv (\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n)^{\text{dual}} = \det \mathbb{A} \tag{24.2}$$

to obtain

$$\boldsymbol{a}_q \wedge \boldsymbol{M}^p = (\delta_q{}^p)^{\text{dual}} = \begin{cases} \text{null } n\text{-form if } p \neq q \\ \text{unit } n\text{-form if } p = q \end{cases} \tag{25}$$

where "null n-form"$\prec 0 \cdot \varepsilon^{i_1 i_2 \ldots i_n}$ and "unit n-form"$\prec 1 \cdot \varepsilon^{i_1 i_2 \ldots i_n} = (1)^{\text{dual}}$. At (25) we have achieved a coordinate-free expression of the generalization of (14). But (25) holds the objects "reciprocal" to the vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$ to be tensors of rank $n-1$ ("psuedo-vectors," or "$(n-1)$-forms" if I may be allowed

[14] See in this connection ELECTRODYNAMICS (1972), p. 184.

[15] Holding $\boldsymbol{A}$ in reserve for use at (26), I adopt the notation $\boldsymbol{M}$ to suggest a row of $\wedge$'s.

a slight misappropriation of language standard to the exterior calculus); if we insist that the objects reciprocal to $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$ be themselves vectors, then we have only to "dualize" the objects in hand, writing

$$\boldsymbol{A}^p \equiv (\boldsymbol{M}^p)^{\text{dual}} \prec A^p{}_j = \tfrac{1}{\det \mathbb{A}}\, \epsilon_{k_1 k_2 \ldots \underset{\substack{\uparrow \\ \text{replace with } j}}{k_p} \ldots k_n}\, a_1^{k_1} a_2^{k_2} \cdots \underset{\substack{\uparrow \\ \text{omit this factor}}}{a_p^{k_p}} \cdots a_n^{k_n} \tag{26}$$

at which point we have recovered precisely (20). Thus far have we proceeded *without appeal to the metric structure* of the vector space; the metric comes into play only when we undertake to "lift the index:" $A^p{}_i \mapsto A^{pi} = g^{ij} A^p{}_j$.

In an effort to further reduce the element of strangeness that (because of the bristling indices?) may still cling to (26), I note that at (14) a bright sophomore might have written

$$\boldsymbol{A}^1 = \tfrac{1}{\det \mathbb{A}} \begin{vmatrix} \boldsymbol{i} & a_2^1 & a_3^1 \\ \boldsymbol{j} & a_2^2 & a_3^2 \\ \boldsymbol{k} & a_2^3 & a_3^3 \end{vmatrix}$$

$$\boldsymbol{A}^2 = \tfrac{1}{\det \mathbb{A}} \begin{vmatrix} a_1^1 & \boldsymbol{i} & a_3^1 \\ a_1^1 & \boldsymbol{j} & a_3^2 \\ a_1^3 & \boldsymbol{k} & a_3^3 \end{vmatrix}$$

$$\boldsymbol{A}^3 = \tfrac{1}{\det \mathbb{A}} \begin{vmatrix} a_1^1 & a_2^1 & \boldsymbol{i} \\ a_1^2 & a_2^2 & \boldsymbol{j} \\ a_1^3 & a_2^3 & \boldsymbol{k} \end{vmatrix}$$

and might, moreover, have noticed that the preceding equations admit straightforwardly of dimensional generalization. Such notation suggests, however, that the results now in hand depend in some critical way upon an orthonormality assumption, which in fact they don't.

5. Reciprocal sets in complex vector spaces. There are actually several distinct ways to "complexify;" I begin by sketching the options in just sufficient detail to indicate which doors I intend to open *en route* to my principal subject matter, and which I will leave shut. So far as concerns notation: I make the adjustment $\boldsymbol{a} \to \boldsymbol{z}$ to lend emphasis to the fact that we work now in complex space, and in place of $\{i, j, \ldots\}$ write $\{\alpha, \beta, \ldots\}$ since I will acquire need also of "dotted indices" $\{\dot{\alpha}, \dot{\beta}, \ldots\}$ and find the distinction between ı and i, ȷ and j unconvincing (besides being awkward to manage in TEX's math mode).

If $\{\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_n\}$ span $\mathcal{C}_n$ then any complex $\boldsymbol{X} \in \mathcal{C}_n$ can be developed $\boldsymbol{X} = X^\alpha \boldsymbol{z}_\alpha$, and an arbitrary change of basis $\boldsymbol{z}_\alpha = T^\beta{}_\alpha \tilde{\boldsymbol{z}}_\beta$ can in this familiar sense

$$\boldsymbol{X} = X^\alpha \boldsymbol{z}_\alpha = X^\alpha (T^\beta{}_\alpha \tilde{\boldsymbol{z}}_\beta) = (T^\alpha{}_\beta X^\beta)\, \tilde{\boldsymbol{z}}_\alpha = \tilde{X}^\alpha \tilde{\boldsymbol{z}}_\alpha$$

be said to induce $X^\alpha \longrightarrow \tilde{X}^\alpha = T^\alpha{}_\beta X^\beta$. Complexification of the preceding remarks inspires first of all the observation that

$$\{\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_n\} \text{ and } \{\bar{\boldsymbol{z}}_1, \bar{\boldsymbol{z}}_2, \ldots, \bar{\boldsymbol{z}}_n\} \text{ are generally distinct}$$

and it would in most contexts be unnatural to assume otherwise; it will always be possible to write $\bar{\boldsymbol{z}}_\alpha = C^\beta{}_\alpha \boldsymbol{z}_\beta$ (though this is seldom done), but would in most contexts be retrograde to assume $C^\alpha{}_\beta = \delta^\alpha{}_\beta$. One must be prepared similarly to allow the elements $T^\alpha{}_\beta$ to be, in general, complex. From this it follows that the elements X^α of a (contravariant) vector and the elements $\bar{X}^\alpha$ of its complex conjugate *transform by generally distinct rules*:

$$
\begin{aligned}
X^\alpha &\longrightarrow \tilde{X}^\alpha = T^\alpha{}_\mu X^\mu \\
\bar{X}^\alpha &\longrightarrow \tilde{\bar{X}}^\alpha = \bar{T}^\alpha{}_\mu \bar{X}^\mu
\end{aligned}
$$

Similarly to be distinguished are the associated covariant rule and its conjugate

$$
\begin{aligned}
\tilde{X}_\beta &\longleftarrow X_\beta = T^\nu{}_\beta \tilde{X}_\nu \quad \text{i.e.,} \quad \tilde{X}_\beta = S^\nu{}_\beta X_\nu \quad \text{with} \quad S^\nu{}_\beta T^\alpha{}_\nu = \delta^\alpha{}_\beta \\
\tilde{\bar{X}}_\beta &\longleftarrow \bar{X}_\beta = \bar{T}^\nu{}_\beta \tilde{\bar{X}}_\nu \quad \text{i.e.,} \quad \tilde{\bar{X}}_\beta = \bar{S}^\nu{}_\beta \bar{X}_\nu
\end{aligned}
$$

Within complex tensor algebra—otherwise known as "spinor algebra," though that term is sometimes reserved for a body of specialized relations[16] which arise *within* complex tensor algebra—one has therefore to distinguish between

- two kinds of covariance, distinguished by dotted/undotted subscripts;
- two kinds of contravariance, . . . dotted/undotted superscripts;[17]
- two kinds of weight (called "weight" and "anti-weight" by some authors[18]).

When we say of the indexed objects $X^{\cdots\alpha_1\cdots\dot{\alpha}_2\cdots}{}_{\ldots\beta_1\ldots\dot{\beta}_2\ldots}$ that they "transform as components of a mixed spinor of weight W and anti-weight M" we mean that their transform can be described[19]

$$
S^W \bar{S}^M \cdots T^{\alpha_1}{}_{\mu_1} \cdots \bar{T}^{\dot{\alpha}_2}{}_{\dot{\mu}_2} \cdots S^{\nu_1}{}_{\beta_1} \cdots \bar{S}^{\dot{\nu}_2}{}_{\dot{\beta}_2} X^{\cdots\mu_1\cdots\dot{\mu}_2\cdots}{}_{\ldots\nu_1\ldots\dot{\nu}_2\ldots}
$$

with $S \equiv \det\|S^\mu{}_\nu\|$. Within such a formalism the numerically invariant Kronecker tensor $\delta^\alpha{}_\beta$ is joined by $\delta^{\dot{\alpha}}{}_{\dot{\beta}}$, which assumes the familiar values, but transforms by the conjugated rule, while the Levi-Civita tensor densities (to the description of which we have now to add the remark that their anti-weights are zero) are joined by $\varepsilon^{\dot{\alpha}_1\dot{\alpha}_2\ldots\dot{\alpha}_n}$ and $\epsilon_{\dot{\beta}_1\dot{\beta}_2\ldots\dot{\beta}_n}$, which have weight $W = 0$ but *anti*-weights given by $M = \pm 1$. I will say of a spinor that it is

$$
\text{of class } \{r, \dot{r}; s, \dot{s}; W, M\}
$$

if it's components display r undotted superscripts, $\dot{r}$ dotted superscripts, s undotted subscripts, $\dot{s}$ dotted subscripts, and if it transforms as a density of weight W and anti-weight M. Generally, it makes transformation-theoretic

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[16] See Élie Cartan, *The Theory of Spinors*, which is the English translation (1966) of a monograph first published in 1937.

[17] This convention is due to B. L. van der Waerden, "Spinoranalyse," Gött. Nachr. 100 (1929), or so I believe.

[18] See E. M. Corson, *Introduction to Tensors, Spinors, and Relativistic Wave Equations* (1953), p. 16.

[19] Compare my QUANTUM MECHANICS (1967), Chapter 2, p. 127.

good sense to speak of *(anti)symmetry with respect to a designated pair of indices* if and only if those indices share the same placement (both up or both down) and are of the same type (both undotted or both dotted); thus

$$X^{\cdots\alpha\cdots\beta\cdots}{}_{\cdots} = X^{\cdots\beta\cdots\alpha\cdots}{}_{\cdots} \text{ makes unrestricted good sense, but}$$
$$X^{\cdots\alpha\cdots}{}_{\cdots\beta\cdots} = X^{\cdots\beta\cdots}{}_{\cdots\alpha\cdots} \text{ does not}$$

The preceding remark is standard to tensor algebra, and carries over directly into spinor algebra. But the availability within the latter formalism of the $*$ operation (complex conjugation) opens the way to some distinctive new possibilities. The conjugate of a spinor is generally a spinor of a different class

$$\{a,b;c,d;e,f\}^* \text{ is of class } \underbrace{\{b,a;d,c;f,e\}}_{\text{original class if and only if } a=b;\,c=d;\,e=f}$$

In connection with spinors of the latter—special—type it becomes possible to speak sensibly of "*hermitian* (anti)symmetry" with respect to designated pairs of similarly placed indices; looking, for example, to the simplest such case $X^{\dot\alpha\beta}$, we have

$$\begin{aligned}\tilde X^{\dot\alpha\beta} = \bar T^{\dot\alpha}{}_{\dot\mu}T^{\beta}{}_{\nu}X^{\dot\mu\nu} \xrightarrow[\text{conjugation}]{} (\tilde X^{\dot\alpha\beta})^* &= T^{\dot\alpha}{}_{\dot\mu}\bar T^{\beta}{}_{\nu}(X^{\dot\mu\nu})^*\\ &\downarrow \text{notational adjustment}\\ \tilde X^{\alpha\dot\beta} &= T^{\alpha}{}_{\mu}\bar T^{\dot\beta}{}_{\dot\nu}X^{\mu\dot\nu}\\ &= \bar T^{\dot\beta}{}_{\dot\mu}T^{\alpha}{}_{\nu}X^{\nu\dot\mu}\end{aligned}$$

from which it follows that

$$\text{if } X^{\dot\mu\nu} = \pm X^{\nu\dot\mu} \quad\text{then}\quad \tilde X^{\dot\alpha\beta} = \pm\tilde X^{\beta\dot\alpha}$$

We will have immediate need of the notion thus introduced.

To lend metric structure to $\mathcal{C}_n$ and, at the same time, to acquire index manipulation capability we (writing $\boldsymbol{z}_\alpha = \boldsymbol{x}_\alpha + i\boldsymbol{y}_\alpha$) introduce

$$\left.\begin{aligned}
g_{\alpha\beta} &\equiv (\boldsymbol{z}_\alpha,\boldsymbol{z}_\beta)\\
&= \big\{(\boldsymbol{x}_\alpha,\boldsymbol{x}_\beta)-(\boldsymbol{y}_\alpha,\boldsymbol{y}_\beta)\big\} + i\big\{(\boldsymbol{x}_\alpha,\boldsymbol{y}_\beta)+(\boldsymbol{y}_\alpha,\boldsymbol{x}_\beta)\big\}\\
&= g_{\beta\alpha} \quad:\quad \text{symmetric square array of complex numbers}\\
g_{\dot\alpha\dot\beta} &\equiv (\bar{\boldsymbol{z}}_\alpha,\bar{\boldsymbol{z}}_\beta)\\
&= \big\{(\boldsymbol{x}_\alpha,\boldsymbol{x}_\beta)-(\boldsymbol{y}_\alpha,\boldsymbol{y}_\beta)\big\} - i\big\{(\boldsymbol{x}_\alpha,\boldsymbol{y}_\beta)+(\boldsymbol{y}_\alpha,\boldsymbol{x}_\beta)\big\}\\
&= g_{\dot\beta\dot\alpha} \quad:\quad \text{complex conjugate of the above}\\
h_{\dot\alpha\beta} &\equiv (\bar{\boldsymbol{z}}_\alpha,\boldsymbol{z}_\beta)\\
&= \big\{(\boldsymbol{x}_\alpha,\boldsymbol{x}_\beta)+(\boldsymbol{y}_\alpha,\boldsymbol{y}_\beta)\big\} + i\big\{(\boldsymbol{x}_\alpha,\boldsymbol{y}_\beta)-(\boldsymbol{y}_\alpha,\boldsymbol{x}_\beta)\big\}\\
&= (h_{\dot\beta\alpha})^* \quad:\quad \text{hermitian square array of complex numbers}\\
&= h_{\beta\dot\alpha}
\end{aligned}\right\} \qquad (27)$$

Those by matrix inversion acquire companions $g^{\alpha\beta}$, $g^{\dot\alpha\dot\beta}$ and $h^{\alpha\dot\beta}$ with the properties

$$\begin{aligned} g^{\alpha\nu}g_{\nu\beta} &= \delta^\alpha{}_\beta \quad \text{and conjugate} \quad g^{\dot\alpha\dot\nu}g_{\dot\nu\dot\beta} = \delta^{\dot\alpha}{}_{\dot\beta} \\ h^{\alpha\dot\nu}h_{\dot\nu\beta} &= \delta^\alpha{}_\beta \quad \text{and conjugate} \quad h^{\dot\alpha\nu}h_{\nu\dot\beta} = \delta^{\dot\alpha}{}_{\dot\beta} \end{aligned}$$

and give rise to a natural generalization of the familiar index manipulation protocol:

$$\begin{aligned} X^{\cdots\ \cdots}_{\cdots\alpha\cdots} &= g_{\alpha\beta}X^{\cdots\beta\cdots}_{\cdots\ \cdots} \quad \text{and} \quad X^{\cdots\alpha\cdots}_{\cdots\ \cdots} = g^{\alpha\beta}X^{\cdots\ \cdots}_{\cdots\beta\cdots} \\ X^{\cdots\ \cdots}_{\cdots\dot\alpha\cdots} &= g_{\dot\alpha\dot\beta}X^{\cdots\dot\beta\cdots}_{\cdots\ \cdots} \quad \text{and} \quad X^{\cdots\dot\alpha\cdots}_{\cdots\ \cdots} = g^{\dot\alpha\dot\beta}X^{\cdots\ \cdots}_{\cdots\dot\beta\cdots} \\ X^{\cdots\ \cdots}_{\cdots\dot\alpha\cdots} &= h_{\dot\alpha\beta}X^{\cdots\beta\cdots}_{\cdots\ \cdots} \quad \text{and} \quad X^{\cdots\alpha\cdots}_{\cdots\ \cdots} = h^{\alpha\dot\beta}X^{\cdots\ \cdots}_{\cdots\dot\beta\cdots} \\ X^{\cdots\ \cdots}_{\cdots\alpha\cdots} &= h_{\alpha\dot\beta}X^{\cdots\dot\beta\cdots}_{\cdots\ \cdots} \quad \text{and} \quad X^{\cdots\dot\alpha\cdots}_{\cdots\ \cdots} = h^{\dot\alpha\beta}X^{\cdots\ \cdots}_{\cdots\beta\cdots} \end{aligned}$$

From (27) we see that the fundamental spinors $g_{\alpha\beta}$ and $h_{\dot\alpha\beta}$ come into coincidence when the basis vectors $\{\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_n\}$ are real (i.e., when $\boldsymbol{y}_\alpha = \mathbf{0} : \alpha = 1, 2, \ldots, n$). But the transformational persistence of such a state of affairs requires the transformation matrix to be real. The distinction between dotted and undotted indices then evaporates; the real and imaginary parts of multiply-indexed complex objects remain transformationally distinct and unmixed, and the theory of spinors degenerates into a duplex copy of the theory of real tensors.

The fundamental spinors $g_{\alpha\beta}$ and $h_{\dot\alpha\beta}$ transform in ways which invite matrix formulation:

$$\begin{aligned} \tilde g_{\alpha\beta} &= T^\mu{}_\alpha g_{\mu\nu}T^\nu{}_\beta \quad \text{can be notated} \quad \tilde{\mathbb{G}} = \mathbb{T}^\intercal\mathbb{G}\mathbb{T} \\ \tilde h_{\dot\alpha\beta} &= \bar T^{\dot\mu}{}_{\dot\alpha} h_{\dot\mu\nu}T^\nu{}_\beta \quad \text{can be notated} \quad \tilde{\mathbb{H}} = \bar{\mathbb{T}}^\intercal\mathbb{H}\mathbb{T} \end{aligned}$$

It becomes clear in the latter notations that imposition of the conditions

$\tilde{\mathbb{G}} = \mathbb{G} = \mathbb{I}$ would force $\mathbb{T}$ to be a *complex rotation matrix*

$\tilde{\mathbb{H}} = \mathbb{H} = \mathbb{I}$ would force $\mathbb{T}$ to be *unitary*

We stand now in possession of all the essential elements needed to construct an account of

- the spinor representations of $O(3)$;
- Pauli spin matrices;
- Dirac spinors;

and other such standard material. I won't, but will instead proceed down a path less traveled: I look to the construction of the basis $\{\boldsymbol{Z}^1, \boldsymbol{Z}^2, \ldots, \boldsymbol{Z}^n\}$ which is "biorthogonal" to a given (generally non-orthogonal) basis $\{\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_n\}$ in $\mathcal{C}_n$. Writing

$$\begin{aligned} \boldsymbol{U} &= U^\alpha \boldsymbol{z}_\alpha \quad \text{whence} \quad \bar{\boldsymbol{U}} = \bar U^{\dot\alpha}\bar{\boldsymbol{z}}_{\dot\alpha} \\ \boldsymbol{V} &= V^\beta \boldsymbol{z}_\beta \end{aligned}$$

we *might* look to the spinor invariant $\boldsymbol{U}\cdot\boldsymbol{V} = U^\alpha g_{\alpha\beta}V^\beta$ but acquire from quantum mechanics a special interest in

$$\bar{\boldsymbol{U}}\cdot\boldsymbol{V} = \bar{U}^{\dot\alpha} h_{\dot\alpha\beta} V^\beta$$

which is distinguished from its companion $\boldsymbol{U}\cdot\boldsymbol{V}$ in this quantum mechanically indispensible respect:

$$\bar{\boldsymbol{U}}\cdot\boldsymbol{U} = \bar{U}^{\dot\alpha} h_{\dot\alpha\beta} U^\beta \text{ is } \begin{cases} \text{manifestly real} \\ \geq 0, \text{ and } = 0 \text{ if and only if } \boldsymbol{U} = \mathbf{0} \end{cases}$$

The latter part of the preceding statement is most familiar in the case $\|h_{\dot\alpha\beta}\| = \mathbb{I}$. Its more general validity hinges on a property (spectral non-negativity) of the hermitian metric, which I now illustrate as it arises in the 2-dimensional case:

$$\|h_{\dot\alpha\beta}\| \equiv \begin{pmatrix} \bar{\boldsymbol{z}}_1\cdot\boldsymbol{z}_1 & \bar{\boldsymbol{z}}_1\cdot\boldsymbol{z}_2 \\ \bar{\boldsymbol{z}}_2\cdot\boldsymbol{z}_1 & \bar{\boldsymbol{z}}_2\cdot\boldsymbol{z}_2 \end{pmatrix} \quad \text{abbreviated} \quad \begin{pmatrix} a & c \\ \bar{c} & b \end{pmatrix} \quad \text{with } a \text{ and } b \text{ real}$$

has eigenvalues

$$\begin{aligned} \lambda &= \tfrac{1}{2}\big\{(a+b) \pm \sqrt{(a+b)^2 - 4(ab - \bar{c}c)}\big\} \\ &= \tfrac{1}{2}\big\{(a+b) \pm \sqrt{(a-b)^2 + 4\bar{c}c}\big\} \end{aligned}$$

But $\bar{c}c \leq ab$ by the Schwarz inequality, and equality is excluded by a linear independence assumption ($\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ span $\mathcal{C}_2$), so

$$= \tfrac{1}{2}\big\{(a+b) \pm (\text{positive number less than } a+b)\big\}$$

I will not linger to develop the more powerful apparatus required to establish such a result in $\mathcal{C}_{n>2}$.

If $\boldsymbol{U} = U^\mu \boldsymbol{z}_\mu$ then $\bar{\boldsymbol{z}}_{\dot\nu}\cdot\boldsymbol{U} = h_{\dot\nu\mu}U^\mu$ gives $h^{\mu\dot\nu}\,\bar{\boldsymbol{z}}_{\dot\nu}\cdot\boldsymbol{U} = U^\mu$ whence

$$\boldsymbol{U} = \boldsymbol{z}_\mu h^{\mu\dot\nu}\bar{\boldsymbol{z}}_{\dot\nu}\cdot\boldsymbol{U} \tag{28}$$

which is the complex analog of (5), and gives back (5) when the basis vectors $\boldsymbol{z}$ are in fact real. We are motivated by the structure of this result to define

$$\left.\begin{aligned} \boldsymbol{Z}^\mu &\equiv h^{\mu\dot\nu}\bar{\boldsymbol{z}}_{\dot\nu} \\ \bar{\boldsymbol{z}}_{\dot\nu} &= h_{\dot\nu\lambda}\boldsymbol{Z}^\lambda \end{aligned}\right\} \tag{29}$$

From $\bar{\boldsymbol{Z}}^{\dot\alpha}\cdot\boldsymbol{Z}^\beta = h^{\dot\alpha\mu}\boldsymbol{z}_\mu\cdot h^{\beta\dot\nu}\bar{\boldsymbol{z}}_{\dot\nu} = h^{\dot\alpha\mu}h_{\mu\dot\nu}h^{\beta\dot\nu}$ we obtain (see again (9))

$$h^{\dot\alpha\beta} = \bar{\boldsymbol{Z}}^{\dot\alpha}\cdot\boldsymbol{Z}^\beta \quad \Longleftarrow \text{compare} \Longrightarrow \quad h_{\dot\alpha\beta} = \bar{\boldsymbol{z}}_{\dot\alpha}\cdot\boldsymbol{z}_\beta \tag{30}$$

The basis $\{\boldsymbol{Z}^1, \boldsymbol{Z}^2, \ldots, \boldsymbol{Z}^n\}$ is "reciprocal/biorthogonal" to $\{\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_n\}$ in this precise sense:

$$\boldsymbol{Z}^\alpha\cdot\boldsymbol{z}_\beta = h^{\alpha\dot\nu}\bar{\boldsymbol{z}}_{\dot\nu}\cdot\boldsymbol{z}_\beta = h^{\alpha\dot\nu}h_{\dot\nu\beta} = \delta^\alpha{}_\beta \tag{31.1}$$

The expansion (28) can in this notation be written

$$\boldsymbol{U} = \boldsymbol{z}_\mu(\boldsymbol{Z}^\mu\boldsymbol{\cdot}\boldsymbol{U}) \tag{31.2}$$

Turning now to the explicit construction of the vectors $\boldsymbol{Z}^\mu$: one could, by mimicry of §2, proceed directly from (29). But the evaluation of $\|h^{\mu\dot\nu}\|$ is tedious except when n is small. It becomes advisable, therefore, to proceed indirectly, by methods which imitate those developed in §4 and exploit the resources of what might be called the "exterior spinor calculus." Such a program is made particularly easy to carry out by the happy circumstance that equations (31) contain no explicit reference conjugated variables, no dotted indices; we (are free, therefore, to make occasional use of roman indices, and) can in direct imitation of (26) write

$$\boldsymbol{Z}^\alpha \equiv (\boldsymbol{M}^\alpha)^{\text{dual}} \prec Z^\alpha{}_\mu = \tfrac{1}{\det\mathbb{Z}}\,\underset{\substack{\uparrow\\ \text{replace with }\mu}}{\epsilon_{k_1k_2\ldots k_\alpha\ldots k_n}}\, z_1^{k_1} z_2^{k_2}\cdots \underset{\substack{\uparrow\\ \text{omit this factor}}}{z_\alpha^{k_\alpha}}\cdots z_n^{k_n} \tag{32}$$

where $\boldsymbol{M}^\alpha$ is a certain $(n-1)^{\text{th}}$-order wedge product—the obvious variant of (24.1)—and where $\det\mathbb{Z} = \epsilon_{k_1k_2\ldots k_n} z_1^{k_1} z_2^{k_2}\cdots z_n^{k_n}$. It is then *obvious* that

$$Z^\alpha{}_\mu z^\mu{}_\beta = \delta^\alpha{}_\beta \tag{33}$$

Note that superscripted $\boldsymbol{z}$'s give rise to subscripted $\boldsymbol{Z}$'s; if we had need of $Z^{\alpha\mu}$ we would have to draw upon $g^{\mu\nu}$, but in fact we appear to have *no* such need.

6. Reciprocal of a system of non-orthogonal functions. Let linearly independent complex-valued functions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ be defined on some interval, which in point merely of notational convenience I will take to be the unit interval $[0,1]$, and let the "inner product" of such functions be defined[20]

$$(\bar f, g) \equiv \int_0^1 \overline{f(x)}g(x)\omega(x)\,dx \quad \text{with } \omega(x) \text{ real and non-negative} \tag{34}$$

We agree to consider those functions to comprise a "natural basis" in an n-dimensional function space $\mathcal{C}_n$, in terms of which the general element can be displayed

$$\varphi(x) = \varphi^\alpha f_\alpha(x)$$

Writing (compare (27))

$$h_{\dot\alpha\beta} \equiv (\bar f_{\dot\alpha}, f_\beta) \tag{35}$$

and proceeding in imitation of (29), we write

$$F^\mu(x) \equiv h^{\mu\dot\nu}\bar f_{\dot\nu}(x) \tag{36}$$

[20] Notice that my notation is non-standard: to achieve conformity with prior practice I write $(\bar f, g)$ where standardly one would write (f,g).

to define the set of functions $\{F^1(x), F^2(x), \ldots, F^n(x)\}$ "reciprocal" to the initial set $\{f_1(x), f_2(x), \ldots, f_n(x)\}$; we expect then to have, as instances of (31.1) and (31.2),

$$(F^\alpha, f_\beta) = \int_0^1 F^\alpha(x) f_\beta(x)\omega(x)\,dx = \delta^\alpha{}_\beta \tag{37.1}$$

$$\varphi(x) = f_\mu(x)\int_0^1 F^\mu(y)\varphi(y)\omega(y)\,dy \tag{37.2}$$

It is instructive—and really the point of this exercise—to consider the *explicit construction* of the functions $F^\mu(x)$. Looking particularly to the case $n = 3$, we have (compare §3 amd note the manifest hermiticity)

$$\mathbb{H}^{-1} = \|h^{\mu\dot\nu}\| = \tfrac{1}{\det\mathbb{H}}\begin{pmatrix} +\begin{vmatrix} h_{\dot 22} & h_{\dot 23} \\ h_{\dot 32} & h_{\dot 33}\end{vmatrix} & -\begin{vmatrix} h_{\dot 12} & h_{\dot 13} \\ h_{\dot 32} & h_{\dot 33}\end{vmatrix} & +\begin{vmatrix} h_{\dot 12} & h_{\dot 13} \\ h_{\dot 22} & h_{\dot 23}\end{vmatrix} \\ -\begin{vmatrix} h_{\dot 21} & h_{\dot 23} \\ h_{\dot 31} & h_{\dot 33}\end{vmatrix} & +\begin{vmatrix} h_{\dot 11} & h_{\dot 13} \\ h_{\dot 31} & h_{\dot 33}\end{vmatrix} & -\begin{vmatrix} h_{\dot 11} & h_{\dot 13} \\ h_{\dot 21} & h_{\dot 23}\end{vmatrix} \\ +\begin{vmatrix} h_{\dot 21} & h_{\dot 22} \\ h_{\dot 31} & h_{\dot 32}\end{vmatrix} & -\begin{vmatrix} h_{\dot 11} & h_{\dot 12} \\ h_{\dot 31} & h_{\dot 32}\end{vmatrix} & +\begin{vmatrix} h_{\dot 11} & h_{\dot 12} \\ h_{\dot 21} & h_{\dot 22}\end{vmatrix} \end{pmatrix} \tag{38}$$

with

$$\begin{aligned}\det\mathbb{H} &= \begin{vmatrix} h_{\dot 11} & h_{\dot 12} & h_{\dot 13} \\ h_{\dot 21} & h_{\dot 22} & h_{\dot 23} \\ h_{\dot 31} & h_{\dot 32} & h_{\dot 33}\end{vmatrix} \equiv h \\ &= \varepsilon^{\beta_1\beta_2\beta_3} h_{\dot 1\beta_1} h_{\dot 2\beta_2} h_{\dot 3\beta_3} \\ &= \int_0^1\!\!\int_0^1\!\!\int_0^1 \big\{\bar f_1(x^1)\bar f_2(x^2)\bar f_3(x^3)\big\}\big\{\varepsilon^{\beta_1\beta_2\beta_3} f_{\beta_1}(x^1) f_{\beta_2}(x^2) f_{\beta_3}(x^3)\big\} \\ &\qquad\qquad\qquad \cdot\,\omega(x^1)\omega(x^2)\omega(x^3)dx^1dx^2dx^3 \\ &= \int_0^1\!\!\int_0^1\!\!\int_0^1 \big\{\bar f_1(x^1)\bar f_2(x^2)\bar f_3(x^3)\big\}\big\{\epsilon_{k_1k_2k_3}\, f_1(x^{k_1}) f_2(x^{k_2}) f_3(x^{k_3})\big\} \\ &\qquad\qquad\qquad \cdot\,\omega(x^1)\omega(x^2)\omega(x^3)dx^1dx^2dx^3\end{aligned}$$

Bringing (38) to (36) we obtain results which can be notated

$$\begin{aligned} F^1(x) &= \tfrac{1}{h}\Big\{+\begin{vmatrix} h_{\dot 22} & h_{\dot 23} \\ h_{\dot 32} & h_{\dot 33}\end{vmatrix}\bar f_1(x) - \begin{vmatrix} h_{\dot 12} & h_{\dot 13} \\ h_{\dot 32} & h_{\dot 33}\end{vmatrix}\bar f_2(x) + \begin{vmatrix} h_{\dot 12} & h_{\dot 13} \\ h_{\dot 22} & h_{\dot 23}\end{vmatrix}\bar f_3(x)\Big\} \\ F^2(x) &= \tfrac{1}{h}\Big\{-\begin{vmatrix} h_{\dot 21} & h_{\dot 23} \\ h_{\dot 31} & h_{\dot 33}\end{vmatrix}\bar f_1(x) + \begin{vmatrix} h_{\dot 11} & h_{\dot 13} \\ h_{\dot 31} & h_{\dot 33}\end{vmatrix}\bar f_2(x) - \begin{vmatrix} h_{\dot 11} & h_{\dot 13} \\ h_{\dot 21} & h_{\dot 23}\end{vmatrix}\bar f_3(x)\Big\} \\ F^3(x) &= \tfrac{1}{h}\Big\{+\begin{vmatrix} h_{\dot 21} & h_{\dot 22} \\ h_{\dot 31} & h_{\dot 32}\end{vmatrix}\bar f_1(x) - \begin{vmatrix} h_{\dot 11} & h_{\dot 12} \\ h_{\dot 31} & h_{\dot 32}\end{vmatrix}\bar f_2(x) + \begin{vmatrix} h_{\dot 11} & h_{\dot 12} \\ h_{\dot 21} & h_{\dot 22}\end{vmatrix}\bar f_3(x)\Big\}\end{aligned}$$

or again

$$F^1(x) = \tfrac{1}{h}\begin{vmatrix} \bar{f}_1(x) & h_{\dot{1}2} & h_{\dot{1}3} \\ \bar{f}_2(x) & h_{\dot{2}2} & h_{\dot{2}3} \\ \bar{f}_3(x) & h_{\dot{3}2} & h_{\dot{3}3} \end{vmatrix}$$

$$F^2(x) = \tfrac{1}{h}\begin{vmatrix} h_{\dot{1}1} & \bar{f}_1(x) & h_{\dot{1}3} \\ h_{\dot{2}1} & \bar{f}_2(x) & h_{\dot{2}3} \\ h_{\dot{3}1} & \bar{f}_3(x) & h_{\dot{3}3} \end{vmatrix}$$

$$F^3(x) = \tfrac{1}{h}\begin{vmatrix} h_{\dot{1}1} & h_{\dot{1}2} & \bar{f}_1(x) \\ h_{\dot{2}1} & h_{\dot{2}2} & \bar{f}_2(x) \\ h_{\dot{3}1} & h_{\dot{3}2} & \bar{f}_3(x) \end{vmatrix}$$

These equations make very clear how it happens that (37.1) has been achieved, for we have

$$\begin{aligned}(F^1, f_\beta) = \tfrac{1}{h}\begin{vmatrix} h_{\dot{1}\beta} & h_{\dot{1}2} & h_{\dot{1}3} \\ h_{\dot{2}\beta} & h_{\dot{2}2} & h_{\dot{2}3} \\ h_{\dot{3}\beta} & h_{\dot{3}2} & h_{\dot{3}3} \end{vmatrix} &= \begin{cases} 1 & \text{if } \beta = 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \delta^1{}_\beta \\ &\ \vdots \\ &\text{etc.}\end{aligned}$$

The dimensional generalization is straightforward.

I have described above a method for constructing functions $F^\alpha(x)$ which are "biorthogonal" to $\{f_1(x), f_2(s), \ldots, f_n(x)\}$ in the sense that

$$\begin{array}{llllll} F^1(x) & \perp & \bullet & f_2(), & f_3(x), & \ldots \quad , f_n(x) \\ F^2(x) & \perp & f_1(x), & \bullet & f_3(x), & \ldots \quad , f_n(x) \\ & \vdots & & & & \\ F^n(x) & \perp & f_1(x), & f_2(x), & f_3(x), & \ldots \qquad \bullet \end{array}$$

That the essence of the method is, in fact, entirely classical becomes clear upon perusal of the §10.1 with which A. Erdélyi *et al*[21] introduce their account of the theory of orthogonal polynomials; the method—applied to a somewhat different objective (construction of a set of functions orthogonal to a given set of linearly independent functions) and used hierarchically

$$\begin{array}{lllll} \phi_2(x) & \perp & f_1(x) & & \\ \phi_3(x) & \perp & f_1(x), & f_2(x) & \\ & \vdots & & & \\ \phi_n(x) & \perp & f_1(x), & f_2(x), & \ldots \quad f_{n-1}(x) \end{array}$$

—has for a long time been familiar as the "Gram–Schmidt orthogonalization process," in which context the expressions $\det \mathbb{H}$ (with matrices $\mathbb{H}$ of ascending

[21] *Higher Transcendental Functions II: Bateman Manuscript Project* (1953).

dimension) are known as "Gram determinants," and play a natural role in the *normalization* of the functions $\phi_k(x)$.[22]

7. Reciprocal of a system of non-orthogonal quantum states. Given a system $\{|\psi_1), |\psi_2), \dots, |\psi_n)\}$ of linearly independent quantum states, we seek a second system $\{|\Psi^1), |\Psi^2), \dots, |\Psi^n)\}$ which is "reciprocal" to the first in the familiar sense

$$|\Psi^\alpha) \perp \{|\psi_1), \dots, |\psi_{\alpha-1}), \underset{\substack{\uparrow \\ \text{omit}}}{|\psi_\alpha)}, |\psi_{\alpha_1}), \dots, |\psi_n)\}$$

of which

$$(\Psi^\alpha|\psi_\beta) = \delta^\alpha{}_\beta$$

provides more compact (and somewhat more detailed) expression. If

$$|\psi) \in \mathcal{H}_n \text{ spanned by } \{|\psi_1), |\psi_2), \dots, |\psi_n)\}$$

we would find ourselves then in position to write

$$|\psi) = \sum_{\alpha=1}^n c^\alpha |\psi_\alpha) \quad \text{with} \quad c^\alpha = (\Psi^\alpha|\psi) \tag{39}$$

"Orthonormality" intrudes spontaneously into quantum mechanical discourse (the eigenstates of observables are orthonormal), but many of the simplifications we have learned to associate with orthonormality—the Fourier decomposition formula (39) is in this resect illustrative—can more properly be attributed to reciprocity or "biorthonormality." The point is seldom remarked because

$$\{|\Psi^1), |\Psi^2), \dots, |\Psi^n)\} \text{ and } \{|\psi_1), |\psi_2), \dots, |\psi_n)\} \text{ become coincident}$$

when the latter happen in fact to be orthonormal; such a state of affairs (for the reason already remarked) often arises spontaneously, but cannot be presumed when $\{|\psi_1), |\psi_2), \dots, |\psi_n)\}$ refers to the states that have been used to concoct a "mixed state."

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[22] Erhard Schmidt (1876–1959) was a student of Hilbert; he is remembered mainly for his study of the integral equation $f(s) = \phi(s) - \lambda \int_a^b K(s,t)\phi(t)\,dt$, which stimulated the development of the "Hilbert space" concept and contributed to the creation of modern functional analysis, of which Schmidt is considered a founding father. The Gram–Schmidt process is described in Schmidt's major paper of 1907. I have, however, been unable to discover any particulars concerning the life and work of the "Gram" who has the distinction of standing in front of the hyphen. For related material, see sections **103.G**, **208.E** and **317.A** in the *Encyclopedic Dictionary of Mathematics* (2nd edition, 1993).

When we refer to a "wave function" $\psi(x)$ we refer in fact not to the quantum state $|\psi)$ itself but to a system of continuously-indexed *coordinates* descriptive of that state:

$$|\psi) = \int |x)\,dx\,\underbrace{(x|\psi)}_{\psi(x)} \quad : \quad \big\{|x)\big\} \text{ are eigenstates of the } \boldsymbol{x}\text{-operator}$$

If we are content to ask for wave functions reciprocal to a system of wave functions

$$\big\{\Psi^1(x), \Psi^2(x), \ldots, \Psi^n(x)\big\} \quad \text{reciprocal to} \quad \big\{\psi_1(x), \psi_2(x), \ldots, \psi_n(x)\big\}$$

—if we are, in other words, content to proceed *in reference to a coordinate system*—then the construction sketched in §6 supplies a detailed answer to the question. And an answer which seems likely to serve most practical needs. But if we insist upon proceeding *without* reference to a coordinate system then we acquire an obligation to undertake formal extension of ideas presented in §4; we need to get in position to write things like

$$|\Psi^\alpha) \sim \big(|\psi_1) \wedge |\psi_2) \wedge \cdots \wedge \underset{\substack{\uparrow\\ \text{omit}}}{|\psi_\alpha)} \wedge \cdots \wedge |\psi_n)\big)^{\text{dual}}$$

I have pursued this topic only far enough to convince myself that one does *not* encounter need of such bizarre objects as "continuously indexed analogs of the Levi-Civita tensor;"[23] the theory appears to unfold without incident (not at all surprisingly, since it does so in every representation), but I am not motivated to pursue it on this occasion.

8. Conclusion. "Reciprocity" in the sense used here—"biorthogonality"—is a useful concept; with its aid one can get along perfectly well even in the absence of orthogonality, doing all the familiar things one is used to relying upon (or so we mistakenly imagine) orthogonality to do. But it seems *not* to be a tool appropriate to the clarification of the issue which motivated this exercise.

[23] One does encounter Levi-Civita tensors, but they wear discrete indices which serve to distinguish one "continuous index" from another.